2-Oct-2017 K-Teoria y K-Teoria equivariante. I ①.

Definition. Let X be a paracompact space. $\mathbb{K} = \mathbb{R}/\mathbb{C}$.
A n-dim. vector bundle $/\mathbb{K}$ is a surjective map $p : E \to X$

1) $p^{-1}(x)$ has a structure of $\mathbb{K}$-vector space.

2) for each $x \in X$, there exists an open neighborhood U of x and a homeom. $\varphi : U \times \mathbb{K}^n \to p^{-1}(U)$

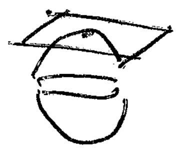
$$\begin{tikzcd} U \times \mathbb{K}^n \arrow[r, "\varphi"] \arrow[dr] & p^{-1}(U) \arrow[d, "p|"] \\ & U \end{tikzcd}$$

Such that $\varphi(u, -) : \mathbb{K}^n \to p^{-1}(u)$ is a linear isomorphism for each u.

Examples

0) $X \times \mathbb{K}^n \to X$.

1) $S^n \hookrightarrow \mathbb{R}^{n+1}$ $E = \{(x,y) \mid y \in S^n, x \in \mathbb{R}^{n+1}, \langle x, y \rangle = 0\}$



$p : E \to S^n$
$(x,y) \mapsto y$.
n-dimensional vector bundle.

2) $TM \to M$ for a C^∞-Manifold.

3) $[0,1] \times \mathbb{R} / (a,b) \sim (c,d)$ $\Leftrightarrow$ $b = -d$ and $\left.\right\}$ $a = 0, c = 1$ / $a = 1, c = 0$

$\mathbb{M} =$ $\pi : \mathbb{M} \to S^1$
$(x,y) \mapsto e^{2\pi i x}$

$S^1 = \underbrace{S^1 \setminus \{1\}}_{U} \cup \underbrace{S^1 \setminus \{-1\}}_{V}$

$\pi^{-1}(U) = (0,1) \times \mathbb{R} \xrightarrow{id} (0,1) \times \mathbb{R}$

$\pi^{-1}(V) = [0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1] \to [0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1] \times$ $\begin{pmatrix} x - \tfrac{1}{2}, y \\ x + \tfrac{1}{2}, y \end{pmatrix}$.

4) $X = \mathbb{C}P^n$, $\{(x, \langle x \rangle)\}$
$x \in \mathbb{C}P^n$, $\xi \xrightarrow{\gamma^n} X$.

5) $X = \mathbb{R}P^n$ $\{(x, \langle x \rangle)\} \to x \in \mathbb{R}P^n$

$S^2 = \mathbb{C}P^1$, $\langle x \rangle \subset \mathbb{C}^2$. $\mathbb{C} \hookrightarrow E \xrightarrow{\eta} \mathbb{C}P^1 = S^2$.

Note: they are contravariant, $p: E \to X$, $f: Y \to X$. $f^*(E)$.

Def. Addition of vector bundles: $p_i : E_i \to X$ ~~for~~ $i = 1, 2$

$$p_1 \times p_2 : E_1 \times E_2 \longrightarrow X \times X, \qquad \Delta: X \longrightarrow X \times X.$$

$$\Delta^*(p_1 \times p_2) : E_1 \oplus E_2 \longrightarrow X.$$

the fiber ~~...~~ is isomorphic to $E_1 \oplus E_2$.

So, one can form. $E_1 \otimes E_2$, E_1^*, $\mathrm{Hom}(E_1; E_2)$

$\Lambda^* E = T(E) / x \otimes x$. $\quad T(E) = \bigoplus_{i=1}^{\infty} (E \otimes_i E)^i$

$\Lambda^* = \bigoplus \Lambda^i \quad \dim \Lambda^i = \binom{n}{i}$.

<u>Examples</u>

1) $E \oplus \mathbb{R} \cong \mathbb{R}^3 \times S^2$ (Example of stably trivial bundle).

2). $M \oplus M \cong \mathbb{R}^2 \times S^1$.

<u>Def</u>. Let X be a compact CW complex.

$\mathrm{Vect}_n^{\mathbb{C}}(X) = \{$ isom. classes of n-dimensional complex VB/X.

$\mathrm{Vect}_n^{\mathbb{R}}(X)$ - Same.

$\mathrm{Vect}^{\mathbb{C}}(X) = \{$isom. of all$\}$, $\mathrm{Vect}^{\mathbb{R}}(X)$.

<u>Comment</u>: There are classifying maps.

$\mathrm{Vect}_n^{\mathbb{R}}(X) \to [X, BO_n]$, and Universal

$\mathrm{Vect}_n^{\mathbb{C}}(X) \to [X, BU_n]$. <u>bundles</u> $\eta : \to BO_n$, $\delta \to BU_n$

X be a compact CW complex.

$KU^0(X) = \text{Groth}(\text{Vect}^{\mathbb{C}}(X))$.

$KO^0(X) = \text{Groth}(\text{Vect}^{\mathbb{R}}(X))$. $\underset{\cong}{\mathbb{N}}$, $\text{Vect}^{\mathbb{C}}(pt)$.

$KU(pt) \cong \mathbb{Z}$ (isom. classes of vector spaces).

$$\widetilde{KU}^0(X) = \ker(KU^0(: X \to pt)).$$

$$KU^0(X) = \mathbb{Z} \oplus \widetilde{KU}^0(X).$$

Analogously KO.

Examples

$\widetilde{KU}^0(S^2) = \mathbb{Z}[x]$. $\widetilde{KO}^0(S^1) = \mathbb{Z}/2$.

$(M \oplus M) = 0$.

$M \neq 0$.

$KU^n(X) := \ker(KU^0(X \times S^n \to X))$.

$KO^n(X) := \ker(KO^0(X \times S^n \to X))$. $\to$ periodic Spectra

How to compute: Atiyah-Hirzebruch Spectral Sequence. Assume you know $H^*(X, \mathbb{Z}/\mathbb{F}_2)$.

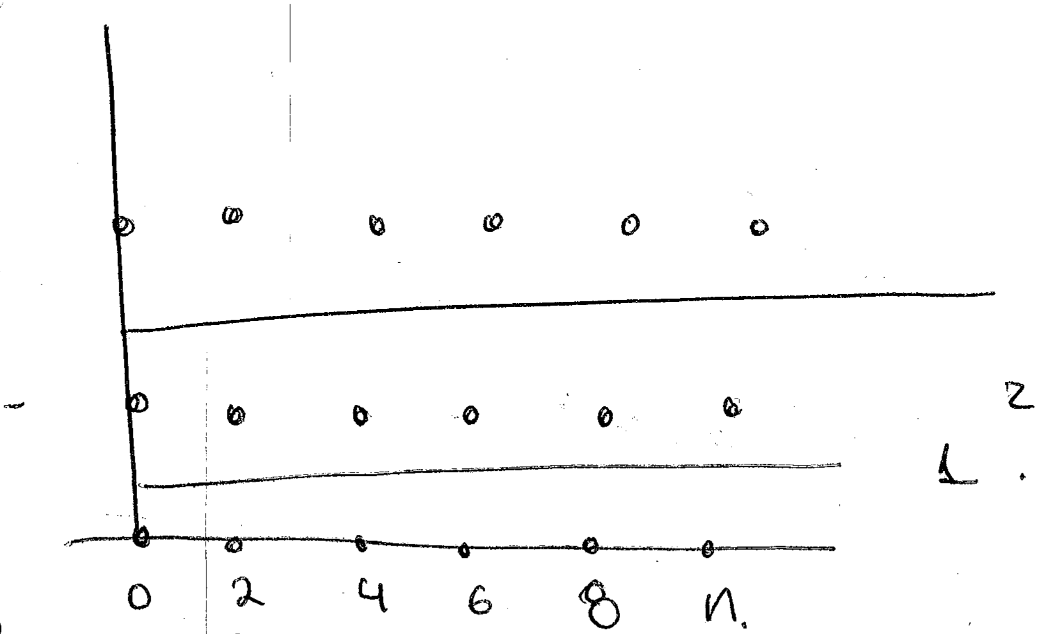
~~Examples~~:

$$E_2^{p,q} = H^p(X, \underline{KU}^q(pt) / \underline{KO}^q(pt)) \Rightarrow KU^*(X) / KO^*(X).$$

Successful cases: for $\overset{KU}{\mathbb{C}P^n}$, $\overset{KO}{S_g}$, $\overset{KO}{}$. $\mathbb{R}P^4$

$$H^*(\mathbb{C}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{even degree}, \\ 0 & \text{otherwise.} \end{cases}$$

$$KU^*(pt) = \begin{cases} \mathbb{Z} & \text{even degree} \\ 0 & \text{otherwise.} \end{cases}$$

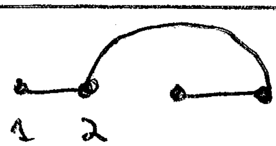


Chessboard pattern. up and down.

No differential $KU^{odd}(\mathbb{C}P^n) = 0$.

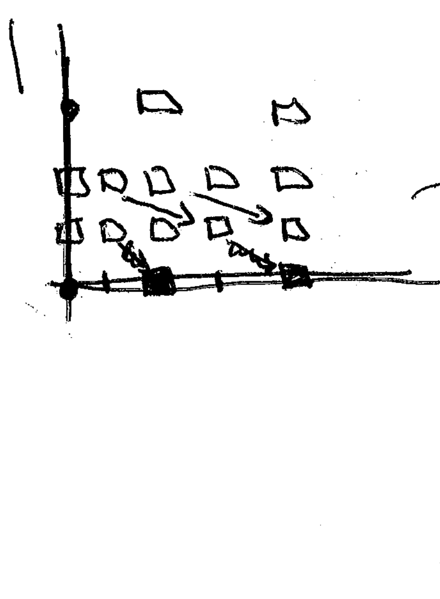
$KU^{2m}(\mathbb{C}P^n) = \bigoplus_{i=0}^{n} \mathbb{Z}$. No d_2, $d_3 = \beta \circ Sq^2$.

$H^*(\mathbb{R}P^4, \mathbb{Z}/2)$:



$\mathbb{Z}$: ~~[illegible]~~.

1	2	3	4
0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$.



$\longrightarrow$ two possibly non-vanishing diff.

Sq^2: $\square$ - $\mathbb{Z}/2$.

$H^2(\mathbb{R}P^2, \mathbb{Z}/2)$ $\bullet$ - $\mathbb{Z}$.

$\downarrow \quad \neq 0$.

$H^4(\mathbb{R}P^2, \mathbb{Z}/2)$.

$\Rightarrow$

and β: $H^1(\mathbb{R}P^2, \mathbb{Z}/2)$.

$\downarrow \quad = 0$.

$H^3(\mathbb{R}P^2, \mathbb{Z}/2)$

<u>Lift grows:</u> G compact lie group. X space ~~G-CW~~. with an action of G. / $= G$ CW.

$\mathrm{Vect}^G_{\mathbb{K}} \overset{\text{Isom.}}{=} \{ E \to X \mid G\text{-}$ vector bundle, $\cancel{G}$-$\}$.

$$\begin{array}{ccc} E_x & \rightsquigarrow & E_{gx} \\ \downarrow & & \downarrow \\ x & & gx \end{array}$$

linear isomorphisms lifting action

$KU^0_G(pt) = \{$ Iso. classes of V complex vector space wit G actn

$KU_G(pt) = R_{\mathbb{C}}(G)$ representation ring.

Example $G = S_3. = \langle x, y \mid x^3, \cancel{y}^2, \cancel{y}x\cancel{y} = x^{-1} \rangle$.

<u>representations:</u> $\underline{id}.$, $\underline{x \to \mathbb{C}}_{\cdot -1}.$, $\underline{D_3 \hookrightarrow Gl(\mathbb{C})}$

<u>Teo</u> $R_{\mathbb{C}}(G) \cong \mathbb{Z}^{(cc(G))}$. as an abelian group.

$R_{\mathbb{C}}(S_3) = \mathbb{Z}[x, y] / x^2 - 1, \cancel{y}^3 - 1$

<u>$KU^n_G(X)$</u> - two periodic, G- cohom. theory.

(Genuine equivariant structure relevant for Mike's purposes, not here, more later!