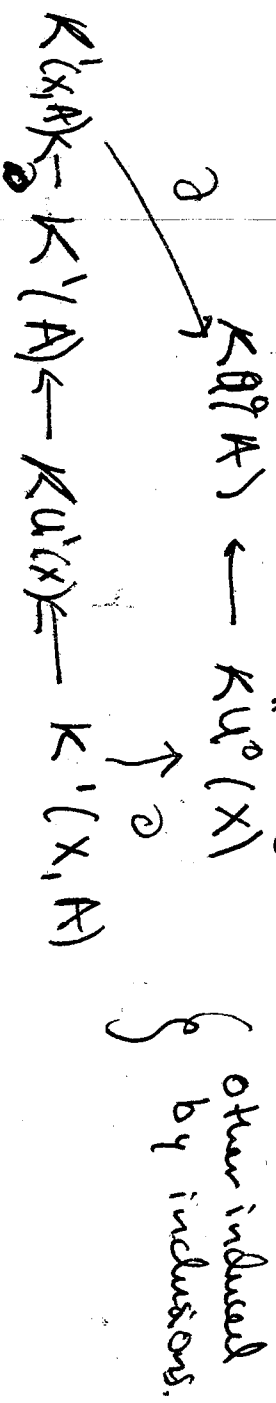


Teoria de Indice Clásica

1. K-Theory is a generalised cohomology theory:
 $t: Y \rightarrow X$ induces $KU^*(Y) \xrightarrow{t^*} KU^*(X)$.

Def ACX closed subspace $K^{-n}(X, A) = K \mathbb{Z}^n(X, C(A))$

$A = \text{cone over } A$, one has a "long" exact sequence

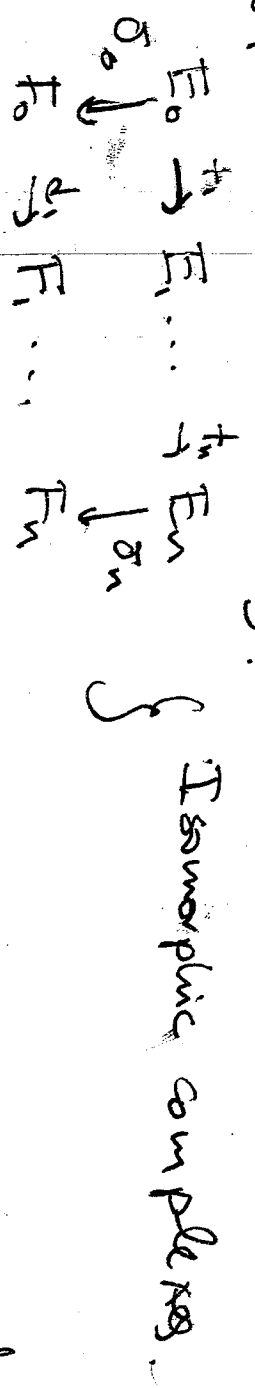


Alternative definition trick:

$$\mathcal{Z}_n \in X, A) = \mathcal{Z}_0 \rightarrow E_0 \xrightarrow{t_1} E_1 \xrightarrow{t_2} \dots \rightarrow E_n \rightarrow 0$$

$E_i \rightarrow X$ vector bundle, $t_i: E_i \rightarrow E_{i-1}$ V.B. morphism,

$t_i \circ t_{i-1}$ exact over $A \mathbb{Z}$.



E is elementary if $E = \mathbb{R}^i \oplus \mathbb{R}^j$, $t_i: \mathbb{R}^i \rightarrow \mathbb{R}^i$ is addition of elementary complexes.

$$\mathcal{Z}_n(X, A) \rightarrow \mathcal{Z}_{n+1}(X, A)$$

$$E_0 \rightarrow \dots \rightarrow E_n \rightarrow 0 \rightarrow (E_0 \rightarrow E_n \rightarrow 0)$$

Lemma $\mathcal{Z}_n(X, A) \rightarrow \mathcal{Z}_{n+1}(X, A)$ is an isomorphism.

$$L(X, A) = \text{column } \mathcal{Z}_n(X, A)$$

~~Prop~~ (Atiyah-Bott-Shapiro).

Prop. There is an equivalence of contravariant functors

$$L(X, A) \rightarrow K^*(X, A).$$

Proof: $IE = (E_0, E_1, +_1) \in L_1(X, A, 1).$

$$X_R = X \times \mathbb{R}K_S \quad K = 0, 1. \quad Z = X_0 \cup X_1.$$

$$0 \rightarrow K(Z, X_1) \rightarrow K(Z) \rightarrow K(X_1) \rightarrow \text{exact.}$$

construct $W \rightarrow Z$ vector bundle, $v: Z \rightarrow X_1$ retraction.

$$W|_{X_R} = E_R, \quad W|_1 = v^*(E, 1).$$

$$[W] - [W] \in \text{Ker } i_* = K^*(Z) \rightarrow K^*(X, 1)$$

\Rightarrow exists $\sigma \in K(X, A)$ $\sigma \mapsto [W] - [W]$

Remarks $K^*(D^n, S^{n-1}) \cong K^*(\mathbb{R}P^n).$

$$(K^*(X) \text{ non compact}) \Rightarrow K^*(X) := K^*(X \cup \text{pt})$$

$$K^*(\mathbb{R}P^n) = K^*(\mathbb{R}P^n \cup \text{pt}) \cong K^*(S^n) \cong K^*(D^n, S^{n-1})$$

Now pick Riem. metric $\langle \cdot, \cdot \rangle: E \times_B E \rightarrow \mathbb{R}$.

$$D(E) = \mathcal{R} \cup \{ \|v\| \leq \Delta \}, \quad \text{for } E \rightarrow X \text{ bundle}$$

We will construct $\lambda_E \in K^0(D(E), S(E)).$

$$0 \rightarrow D \times D E \xrightarrow{\lambda} E \oplus \lambda^* E \xrightarrow{\lambda} \mathbb{R}^2 E \dots \rightarrow \mathbb{R}^{\dim E} \rightarrow 0$$



$$D \xrightarrow{\pi} D = M$$



$$\text{is } K(D, S^1) \cong$$

(3)

$$\beta \in K^0(S^2) \cong K^0(\text{point})$$

Is the generator 1

Definition: The Thom homomorphism:

$$K^*(X) \rightarrow K^*(D(E)^0) \xrightarrow{\cdot \chi_E} K^*(DE, SE)$$

(E is a \mathbb{Z}_2 -Spinor bundle, more on this later).

Atiyah - Singer

Construction - ~~Topological~~ ~~pushforward~~

Let $X \subset Y$ be a pair of smooth manifolds, X compact. $N \supset X$ tubular neighborhood.

$N \cong \delta(x \rightarrow y)$. $TX \subset TY$ submanifold.

$TN \supset$ Tubular neighborhood of TX in TY .

$TN \cong$ vector bundle over TX by lifting $N \oplus N$.

$$K^*(TX) \xrightarrow{\text{Ann}} K^*(TN) \xrightarrow{N \otimes_{\mathbb{R}} D}$$

$$\downarrow i \quad \downarrow i$$

$$K^*(TX) \quad K^*(TY)$$

~~retract~~
 TN open in TY
 retract $TY \rightarrow TN$

$$i^* \circ i_! : K^*(TX) \rightarrow K^*(TX)$$

$$\cong \sum (-1)^i \chi_i(N \otimes_{\mathbb{R}} D) \cdot$$

Topological Index

X smooth manifold. $X \hookrightarrow \mathbb{R}^{2n}$ $j: \text{pt} \rightarrow \mathbb{R}^{2n}$

$$K^0(TX) \xrightarrow{i!} K^0(T\mathbb{R}^{2n}) \xleftarrow{j!} K^0(\text{pt})$$

Top. Index

$$K^0(X) \xrightarrow{\quad} \mathbb{Z} \cdot \text{Index}^{(\text{top})}$$

Split gears: Analysis

E smooth bundle.

$$D: \Gamma(E) \rightarrow \Gamma(E)$$

Differential operator. n given on coordinates

$$D = \sum_{\alpha} A_{\alpha} \frac{\partial}{\partial x^{\alpha}}$$

A_{α} - matrix $n \times n$ with \mathbb{C} coefficients on \mathbb{R}^n

E.g.:

The symbol is A_{α} of highest order. It is elliptic if $\langle A_{\alpha} v, v \rangle \geq M \langle v, v \rangle$.

\Rightarrow iso outside $0!$. $\Rightarrow \sigma(D): E \rightarrow E$.

is an iso over $S(E)$.

$$\Rightarrow \sigma(D) \in \mathcal{K}(D(E), S(E))$$

An index = $\dim \ker D - \dim \text{coker } D \in \mathbb{Z}$.

Atiyah-Singer index Theorem:

An index = Top. Index.

And Cohomological formula:

An. index = $ch(\sigma(D)) \cdot TD(M) [M]$

Ch - Chern character, $TD(M)$ - Todd Class, Polyno-
mial in Chern classes.

Interesting operators: M - manifolds.

$\Omega^*(M)$ - Hodge - de Rham Operator.

Index $\stackrel{\text{interesting}}{=} \chi(M)$

If $E \rightarrow M$ is a n dimensional Bundle (get it
 M is spin, one can form.

D - Dirac - Atiyah-Singer op.

$\text{Index}(D) \in KO^k(\text{pt})$.

(Clifford index), \tilde{A} .

• The index is a bar dism invariant,
 Ω^{spin} \rightarrow KO . Spectra map.

Ω^{Spin} \rightarrow KU .