Teoria de Índice Clásica

1. K-Theory is a generalized cohomology theory: $f: X \to Y$ induces $KU^*(Y) \xrightarrow[f^*]{} KU^*(X)$.

<u>Def</u> $A \subset X$ closed subspace $K^{-n}(X,A) = KU^{n}(X \cup_A CA)$

A = Cone over A, one has a "long" exact sequence

$$\begin{array}{ccc} KU^0(A) & \leftarrow & KU^0(X) \\ & & \uparrow \partial \\ K^1(X,A) \xleftarrow{\partial} K^1(A) \leftarrow KU^1(X) & \leftarrow & K^1(X,A) \end{array}$$

other induced by inclusions.

A lternative definition trick:

$$\mathcal{L}_n(X,A) = \{0 \to E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} E_n \to 0 \mid$$

$E_i \to X$ vector bundle, $f_i : E_{i-1} \to E_i$ V.B. morphism, and $f_i \circ f_{i-1}$ exact over $A\}$.

$$\begin{array}{ccc} E_0 \xrightarrow{f_1} E_1 \dots & \xrightarrow{f_n} & E_n \\ \sigma_0 \downarrow & & \downarrow \sigma_n \\ F_0 \xrightarrow{d_1} F_1 \dots & & F_n \end{array}$$

Isomorphic complexes.

$\mathbb{E}$ is elementary if $E_i = E_{i-1}$, $f_i : E_{i-1} \xrightarrow{id} E_i$

There is addition $\oplus$, $L_n = \mathcal{L}_n /$ addition of elementary complexes.

$\mathbb{E} \oplus E_1 \oplus \dots E_k \cong \mathbb{F} \oplus \dots F_1 \oplus \dots F_k$.

$\mathcal{L}_n(X,A) \to \mathcal{L}_{n+1}(X,A)$.

$E_0 \to \dots E_n \longmapsto (E_0 \to E_n \to 0)$.

<u>Lemma</u> $L_n(X,A) \to L_{n+1}$ is an isomorphism.

$L(X,A) := \operatorname{colim} L_n(X,A)$.

~~Prop~~ (Atiyah–Bott–Shapiro).

Prop. There is an equivalence of contravariant functors
$$L(X,A) \longrightarrow K(X,A).$$

Proof: $\mathbb{E} = (E_0, E_1, t_1) \in L_1(X,A)$.

$X_k = X \times \{k\}$ $k = 0,1$. $Z = X_0 \cup_A X_1$.

$$0 \to K(Z, X_1) \to K(Z) \to K(X_1) \to \text{ split exact.}$$

$r: Z \to X_1$ retraction

construct $W \mapsto Z$ vector bundle,

$W|_{X_k} = E_k$, $W_1 := r^*(E_1)$.

$[W] - [W_1] \in \ker i^* . K^*(Z) \to K^*(X_1)$

$\Rightarrow$ exists $\sigma \in K(X,A)$ $\sigma \mapsto [W] - [W_1]$

Remarks $K^*(D^n, S^{n-1}) \cong K^*(\mathbb{R}^n)$.

($K^*(X)$ X noncompact $\Rightarrow K^*(X) := K^*(X \amalg pt)$)

$K^*(\mathbb{R}^n) = K^*(\mathbb{R}^n \cup \{\infty\}) \cong K^*(S^n) \cong K^*(D^n, S^{n-1})$!

Now pick Riem. metric $\langle\,\rangle : E \times_B E \to \mathbb{C} / \mathbb{R}$.

$D(E) = \{ v \mid \|v\| \leq 1 \}$, for $E \to X$ bundle

We will construct $\lambda_E \in K^0(DE, SE)$.

$$0 \to \mathbb{C} \times DE \xrightarrow{\wedge v} E \doteq \Lambda^1 E \xrightarrow{\wedge v} \Lambda^2 E \dots \to \Lambda^{\dim} E \to 0$$

DE.

e.g.: $\begin{tikzcd} \mathbb{C} \arrow[r, "\lambda^\vee"] \arrow[dr] & \mathbb{C} = \lambda\mathbb{C} \arrow[d] \\ & pt \end{tikzcd}$ is $K(D\mathbb{C}, S(\mathbb{C})) \cong$

$\beta \in K^0(S^2) \cong K^0(\mathbb{C} \cup pt)$.

Is the Bott generator!

Definition: The Thom homomorphism:

$$K^*(X) \to K^*(D(E)^\circ) \xrightarrow{\cdot \lambda_E \cdot} K^*(DE, SE).$$

(Iso if E - Spin(c) bundle, more on this later).

Construction - ~~topological index~~ Atiyah-Singer pushforward

Let $X \subset Y$ be a pair of smooth manifolds, X compact. $N \supset X$ tubular neighborhood.

$N \approx \nu(i : X \to Y)$. $TX \subset TY$ submanifold.

TN, Tubular neighborhood of TX in TY.

$TN \approx$ vector bundle over TX by lifting $N \oplus N$

$N \oplus iN$, $TN \approx \underline{\pi^*(N \otimes_{\mathbb{R}} \mathbb{C})}$

$$\begin{tikzcd} K^*(TX) \arrow[r, "\lambda_{TN}"] \arrow[dr, "i_!"'] & K^*(TN) \arrow[d] \\ & K^*(TY) \end{tikzcd}$$

(~~vector bundle over~~ TN open in TY | retract $TY \xrightarrow{r} TN$).

$i^* \circ i_! : K^*(TX) \to K^*(TX)$.

$$\sum (-1)^i \lambda^i (N \otimes_{\mathbb{R}} \mathbb{C}).$$

opological Index

X smooth manifold. $X \overset{i}{\hookrightarrow} \mathbb{R}^{2n}$

$j : pt \to \mathbb{R}^{2n}$.

$$\begin{array}{ccccc} K^0(TX) & \xrightarrow{i_!} & K^0(T\mathbb{R}^{2n}) & \xleftarrow{j_!} & K^0(T pt) \\ & & & & \cong \\ & & & & \mathbb{Z}. \end{array}$$

Top. Index

$K^0(\cancel{M}) \longrightarrow \mathbb{Z}$. (top) Index.

Shift gears: Analysis.

E smooth complex bundle. $D : \Gamma(E) \longrightarrow \Gamma(E)$.

Differential operator. of order n. given on coordinates

$D = \sum A_\alpha \dfrac{\partial^{|\alpha|}}{\partial^\alpha}$ α multi index A_α - matrix $n \times n$ with coef. on C^∞

E.g:

The Symbol is A_α of highest order.

It is elliptic if $\langle A_\alpha v, v \rangle \geq M \langle v, v \rangle$.

$\Rightarrow$ iso outside 0! $\Rightarrow \sigma(D) : E \to E$.

is an iso over $S(E)$.

$\Rightarrow \sigma(D) \in K(D(E), S(E))$.

An index $=$ dim ker. D - dim coker D. $\in \mathbb{Z}$.

Atiyah-Singer index Theorem:

An index = Top. Index.

And Cohomological formula:

An. index = $ch(\sigma(D)) \cdot Td(M)\,[M]$.

Ch - chern character, Td(M) - Todd Class, polynomial in chern Classes.

Interesting operators: M - manifolds.

$\Omega^*(M)$ - Hodge-deRham Operator.

Index $\underset{\text{interesting}}{=} \chi(M)$

If $E \to M$ is a "Clifford" Bundle (get it M is spin, One can form.

$\not{D}$ - "dirac" - Atiyah-Singer op.

$\mathrm{Index}(\not{D}) \in KO^k(pt)$.

(Clifford index), $\hat{A}$.

- The index is a bordism invariant,

$\Omega^{spin}_{*} \longrightarrow KO$. Spectra map.

$\Omega^{spin(c)} \longrightarrow KU$.